Chapter
Differential Equations

Topic-1: Ordinary Differential Equations, Order & Degree of Differential Equations

1 MCQs with One Correct Answer

1. If $x^2+y^2=1$, then [2000S]
(a) $yy''-2(y')^2+1=0$ (b) $yy''+(y')^2+1=0$
(c) $yy''+(y')^2-1=0$ (d) $yy''+2(y')^2+1=0$

6 MCQs with One or More than One Correct Answer

2. Consider the family of all circles whose centers lie on the straight line $y=x$. If this family of circle is represented by the differential equation $Py''+Qy'+1=0$, where P, Q are functions of x, y and y' $\left(\text{here } y'=\frac{dy}{dx}, y''=\frac{d^2y}{dx^2}\right)$, then which of the following statements is (are) true? [Adv. 2015]
(a) $P=y+x$ (b) $P=y-x$
(c) $P+Q=1-x+y+y'+(y')^2$ (d) $P-Q=x+y-y'-(y')^2$

3. A curve passes through the point $\left(1,\frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x}+\sec\left(\frac{y}{x}\right), x>0.$ Then the equation of the curve is [Adv. 2013]
(a) $\sin\left(\frac{y}{x}\right)=\log x+\frac{1}{2}$ (b) $\operatorname{cosec}\left(\frac{y}{x}\right)=\log x+2$
(c) $\sec\left(\frac{2y}{x}\right)=\log x+2$ (d) $\cos\left(\frac{2y}{x}\right)=\log x+\frac{1}{2}$

4. The differential equation representing the family of curves $y^2=2c\left(x+\sqrt{c}\right)$, where c is a positive parameter, is of [1999 – 3 Marks]
(a) order 1 (b) order 2 (c) degree 3 (d) degree 4

5. The order of the differential equation whose general solution is given by $y=(c_1+c_2)\cos(x+c_3)-c_4e^{x+c_5}$, where c_1, c_2, c_3, c_4, c_5, are arbitrary constants, is [1998 – 2 Marks]
(a) 5 (b) 4 (c) 3 (d) 2

10 Subjective Problems

6. A normal is drawn at a point $P(x, y)$ of a curve. It meets the x–axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is
$$y\frac{dy}{dx}=\pm\sqrt{k^2-y^2}$$ [1994 – 5 Marks]
Find the equation of such a curve passing through $(0, k)$.

7. If $(a+bx)e^{y/x}=x$, then prove that $x^3\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^2$ [1983 – 3 Marks]

Topic-2: General & Particular Solution of Differential Equation

1 MCQs with One Correct Answer

1. If $y=y(x)$ satisfies the differential equation $8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy=\left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1}dx, x>0$ and $y(0)=\sqrt{7}$, then $y(256)=$ [Adv. 2018]
(a) 3 (b) 9 (c) 16 (d) 80

2. The differential equation $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{y}$ determines a family of circles with [2005S]
(a) variable radii and a fixed centre at (0, 1)
(b) variable radii and a fixed centre at (0, –1)
(c) fixed radius 1 and variable centres along the x–axis.
(d) fixed radius 1 and variable centres along the y–axis.

3. For the primitive integral equation $ydx + y^2dy = x\,dy$; $x \in R, y > 0, y = y(x), y(1) = 1$, then $y(-3)$ is [2005S]
(a) 3 (b) 2 (c) 1 (d) 5

4. The solution of primitive integral equation $(x^2 + y^2)\,dy = xy\,dx$ is $y = y(x)$. If $y(1) = 1$ and $(x_0) = e$, then x_0 is equal to [2005S]
(a) $\sqrt{2(e^2-1)}$ (b) $\sqrt{2(e^2+1)}$
(c) $\sqrt{3}\,e$ (d) $\sqrt{\frac{e^2+1}{2}}$

5. If $y = y(x)$ and it follows the relation $x\cos y + y\cos x = \pi$ then $y''(0) =$ [2005S]
(a) 1 (b) –1 (c) $\pi - 1$ (d) $-\pi$

6. If $y = y(x)$ and $\frac{2+\sin x}{y+1}\left(\frac{dy}{dx}\right) = -\cos x, y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ equals [2004S]
(a) 1/3 (b) 2/3 (c) –1/3 (d) 1

7. If $y(t)$ is a solution of $(1+t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$, then $y(1)$ is equal to [2003S]
(a) –1/2 (b) $e + 1/2$
(c) $e - 1/2$ (d) 1/2

8. A solution of the differential equation [1999 – 2 Marks]
$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$ is
(a) $y = 2$ (b) $y = 2x$
(c) $y = 2x - 4$ (d) $y = 2x^2 - 4$

2 Integer Value Answer/ Non-Negative Integer

9. If $y(x)$ is the solution of the differential equation $xdy - (y^2 - 4y)dx = 0$ for $x > 0$, $y(1) = 2$, and the slope of the curve $y = y(x)$ is never zero, then the value of $10\,y(\sqrt{2})$ is ____________. [Adv. 2022]

10. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation
$f(x+y) = f(x)f'(y) + f'(x)f(y)$ for all $x, y \in \mathbb{R}$
Then, the value of $\log_e(f(4))$ is _____. [Adv. 2018]

3 Numeric/ New Stem Based Questions

11. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation
$\frac{dy}{dx} = (2 + 5y)(5y - 2)$,
then the value of $\lim_{x \to -\infty} f(x)$ is _____. [Adv. 2018]

6 MCQs with One or More than One Correct Answer

12. Let Γ denote a curve $y = y(x)$ which is in the first quadrant and let the point (1, 0) lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_p. If PY_p has length 1 for each point P on Γ, then which of the following options is/are correct? [Adv. 2019]
(a) $y = -\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$
(b) $xy' - \sqrt{1-x^2} = 0$
(c) $y = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$
(d) $xy' + \sqrt{1-x^2} = 0$

13. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0, x > 0$, passes through the point (1, 3). Then the solution curve [Adv. 2016]
(a) intersects $y = x + 2$ exactly at one point
(b) intersects $y = x + 2$ exactly at two points
(c) intersects $y = (x + 2)^2$
(d) does NOT intersect $y = (x + 3)^2$

14. Let $y(x)$ be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statement is (are) true? [Adv. 2015]
(a) $y(-4) = 0$ (b) $y(-2) = 0$
(c) $y(x)$ has a critical point in the interval (–1, 0)
(d) $y(x)$ has no critical point in the interval (–1, 0)

9 Assertion and Reason /Statement Type Questions

15. Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$ satisfy $y(2) = \frac{2}{\sqrt{3}}$.
STATEMENT-1 : $y(x) = \sec\left(\sec^{-1}x - \frac{\pi}{6}\right)$ and
STATEMENT-2 : $y(x)$ is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$ [2008]
(a) **STATEMENT - 1** is True, STATEMENT - 2 is True; **STATEMENT- 2 is a correct explanation for** STATEMENT - 1

(b) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT-2 is NOT a correct explanation for STATEMENT - 1
(c) STATEMENT - 1 is True, STATEMENT - 2 is False
(d) STATEMENT - 1 is False, STATEMENT - 2 is True

10 Subjective Problems

16. If length of tangent at any point on the curve $y = f(x)$ intecepted between the point and the x–axis is of length 1. Find the equation of the curve. [2005 – 4 Marks]

17. A hemispherical tank of radius 2 metres is initially full of water and has an outlet of 12 cm^2 cross-sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $v(t) = 0.6\sqrt{3gh(t)}$, where $v(t)$ and $h(t)$ are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time t, and g is the acceleration due to gravity. Find the time it takes to empty the tank. (Hint : Form a differential equation by relating the decrease of water level to the outflow). [2001 – 10 Marks]

18. A country has a food deficit of 10%. Its population grows continously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self–sufficient in food after n years, where n is the smallest integer bigger than or equal to $\frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$. [2000 – 10 Marks]

19. A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x–axis. Determine the equation of the curve. [1999 – 10 Marks]

20. Determine the equation of the curve passing through the origin, in the form $y = f(x)$, which satisfies the differential equation $\frac{dy}{dx} = \sin(10x + 6y)$. [1996 – 5 Marks]

Topic-3: Linear Differential Equation of First Order

1 MCQs with One Correct Answer

1. Let $f : [1, \infty) \to \mathbb{R}$ be a differentiable function such that $f(1) = \frac{1}{3}$ and $3\int_1^x f(t)\,dt = x f(x) - \frac{x^3}{3}, x \in [1, \infty)$ Let e denote the base of the natural logarithm. Then the value of f (e) is [Adv. 2023]
(a) $\frac{e^2+4}{3}$ (b) $\frac{\log_e 4 + e}{3}$
(c) $\frac{4e^2}{3}$ (d) $\frac{e^2-4}{3}$

2. The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$ in (– 1, 1) satisfying $f(0) = 0$. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)\,d(x)$ is [Adv. 2014]
(a) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (b) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$
(c) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (d) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

2 Integer Value Answer/ Non-Negative Integer

3. For $x \in \mathbb{R}$, let y(x) be a solution of the differential equation $(x^2-5)\frac{dy}{dx} - 2xy = -2x(x^2-5)^2$ such that y(2) = 7. Then the maximum value of the function y(x) is [Adv. 2023]

4. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0, x \in \boldsymbol{R}$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then the value of $y(2)$ is [2011]

6 MCQs with One or More than One Correct Answer

5. For $x \in \mathbb{R}$, let the function y(x) be the solution of the differential equation $\frac{dy}{dx} + 12y = \cos\left(\frac{\pi}{12}x\right)$, $y(0) = 0$
Then, which of the following statements is/are TRUE ? [Adv. 2022]
(a) $y(x)$ is an increasing function
(b) $y(x)$ is a decreasing function
(c) There exists a real number β such that the line $y = \beta$ intersects the curve $y = y(x)$ at infinitely many points
(d) $y(x)$ is a periodic function

6. For any real numbers α and β, let $y_{\alpha,\beta}(x), x \in \mathbb{R}$, be the solution of the differential equation
$$\frac{dy}{dx} + \alpha y = xe^{\beta x}, y(1) = 1$$

Let $S=\{y_{\alpha,\beta}(x):\alpha,\beta\in\mathbb{R}\}$. Then which of the following functions belong(s) to the set S? [Adv. 2021]

(a) $f(x)=\frac{x^2}{2}e^{-x}+\left(e-\frac{1}{2}\right)e^{-x}$

(b) $f(x)=-\frac{x^2}{2}e^{-x}+\left(e+\frac{1}{2}\right)e^{-x}$

(c) $f(x)=\frac{e^x}{2}\left(x-\frac{1}{2}\right)+\left(e-\frac{e^2}{2}\right)e^{-x}$

(d) $f(x)=\frac{e^x}{2}\left(\frac{1}{2}-x\right)+\left(e+\frac{e^2}{2}\right)e^{-x}$

7. Let $f:[0,\infty)\to\mathbb{R}$ be a continuous function such that

$$f(x)=1-2x+\int_0^x e^{x-t}f(t)dt$$

for all $x\in[0,\infty)$. Then, which of the following statement (s) is (are) TRUE? [Adv. 2018]

(a) The curve $y=f(x)$ passes through the point (1, 2)

(b) The curve $y=f(x)$ passes through the point (2, –1)

(c) The area of the region $\left\{(x,y)\in[0,1]\times\mathbb{R}:f(x)\le y\le\sqrt{1-x^2}\right\}$ is $\frac{\pi-2}{4}$

(d) The area of the region $\left\{(x,y)\}\in[0,1]\times\mathbb{R}:f(x)\le y\le\sqrt{1-x^2}\right\}$ is $\frac{\pi-1}{4}$

8. Let f : $(0,\infty)\to\mathbb{R}$ be a differentiable function such that $f'(x)=2-\frac{f(x)}{x}$ for all $x\in(0,\infty)$ and $f(1)\neq 1$. Then [Adv. 2016]

(a) $\lim_{x\to 0+} f'\left(\frac{1}{x}\right)=1$

(b) $\lim_{x\to 0+} xf\left(\frac{1}{x}\right)=2$

(c) $\lim_{x\to 0+} x^2 f'(x)=0$

(d) $|f(x)|\le 2$ for all $x\in(0,2)$

9. If $y(x)$ satisfies the differential equation $y'-y\tan x=2x\sec x$ and $y(0)=0$, then [2012]

(a) $y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{8\sqrt{2}}$

(b) $y'\left(\frac{\pi}{4}\right)=\frac{\pi^2}{18}$

(c) $y\left(\frac{\pi}{3}\right)=\frac{\pi^2}{9}$

(d) $y'\left(\frac{\pi}{3}\right)=\frac{4\pi}{3}+\frac{2\pi^2}{3\sqrt{3}}$

7 Match the Following

10. Match the following : [2006 - 6M]

Column I	Column II
(A) $\int_0^{\pi/2}(\sin x)^{\cos x}\left(\cos x\cot x-\log(\sin x)^{\sin x}\right)dx$	(p) 1
(B) Area bounded by $-4y^2=x$ and $x-1=-5y^2$	(q) 0
(C) Cosine of the angle of intersection of curves $y=3^{x-1}\log x$ and $y=x^x-1$ is	(r) 6 ln 2
(D) Let $\frac{dy}{dx}=\frac{6}{x+y}$ where $y(0)=0$ then value of y when $x+y=6$ is	(s) $\frac{4}{3}$

Answer Key

Topic-1 : Ordinary Differential Equations, Order & Degree of Differential Equations

1. (b) 2. (b,c) 3. (a) 4. (a,c) 5. (c)

Topic-2 : General & Particular Solution of Differential Equation

1. (a) 2. (c) 3. (a) 4. (c) 5. (c) 6. (a) 7. (a) 8. (c) 9. (8)
10. (2) 11. (0.4) 12. (c,d) 13. (a,d) 14. (a,c) 15. (c)

Topic-3 : Linear Differential Equation of First Order

1. (c) 2. (b) 3. (16) 4. (0) 5. (c) 6. (a,c) 7. (b,c) 8. (a) 9. (a, d)
10. (A)→p, (B)→s, (C)→p, (D)→r

Hints & Solutions

Topic-1: Ordinary Differential Equations, Order & Degree of Differential Equations

1. (b) Given $x^2 + y^2 = 1$. Differentiating w.r.t. x, we get

$x + yy' = 0$

Again differentiating w.r.t. x,

$1 + (y')^2 + yy'' = 0$

2. (b, c) Let the equation of circle be

$x^2 + y^2 + 2gx + 2gy + c = 0$

$\Rightarrow\ 2x + 2yy' + 2g + 2gy' = 0$

$\Rightarrow\ x + yy' + g + gy' = 0$ ...(i)

On differentiating again, we get

$$1 + yy'' + (y')^2 + gy'' = 0 \Rightarrow g = -\left[\frac{1+(y')^2 + yy''}{y''}\right]$$

On substituting the value of g in eqn. (i), we get

$$x + yy' - \frac{1+(y')^2 + yy''}{y''} - \left(\frac{1+(y')^2 + yy''}{y''}\right)y' = 0$$

$\Rightarrow\ xy'' + yy'y'' - 1 - (y')^2 - yy'' - y' - (y')^3 - yy'y'' = 0$

$\Rightarrow\ (x - y)y'' - y'(1 + y' + (y')^2) = 1$

$\Rightarrow\ (y - x)y'' + [1 + y' + (y')^2]\,y' + 1 = 0$

$\because\ Py'' + Qy' + 1 = 0$

$\therefore\ P = y - x,\ Q = 1 + y' + (y')^2$

Hence, $P + Q = 1 - x + y + y' + (y')^2$

3. (a) $\dfrac{dy}{dx} = \dfrac{y}{x} + \sec\dfrac{y}{x}$

Put $y = vx$, $\therefore \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\Rightarrow\ x\dfrac{dv}{dx} = \sec v \Rightarrow \displaystyle\int \cos v\, dv = \int \frac{dx}{x}$

$\Rightarrow\ \sin v = \log x + c$ $(\because x > 0)$

$\Rightarrow\ \sin\dfrac{y}{x} = \log x + c$

Since, it passes through $\left(1, \dfrac{\pi}{6}\right) \Rightarrow c = \dfrac{1}{2}$

Hence, $\sin\dfrac{y}{x} = \log x + \dfrac{1}{2}$

4. (a, c) $y^2 = 2c(x + \sqrt{c}) \Rightarrow 2yy_1 = 2c \Rightarrow c = yy_1$

Eliminating c, we get

$$y^2 = 2yy_1(x + \sqrt{yy_1}) \text{ or } (y - 2xy_1)^2 = 4yy_1^3$$

It involves only Ist order derivative, its order is 1 but its degree is 3 as y_1^3 is there.

5. (c) The given solution of differential equation is

$y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x + c_5}$

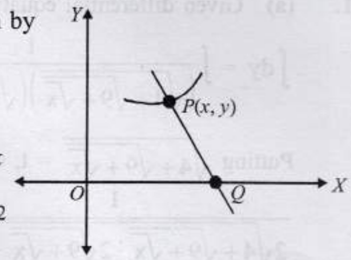
$= (c_1 + c_2)\cos(x + c_3) - c_4 . e^{c_5} . e^x$

$= A\cos(x + c_3) - Be^x$

[Here, $c_1 + c_2 = A, c_4 e^{c_5} = B$]

Hence in the solution there are actually three arbitrary constants and hence this differential equation should be of order 3.

6. The length of normla PQ to any curve $y = f(x)$ is given by



$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Given, length of $PQ = k$

$\therefore\ \left(y\dfrac{dy}{dx}\right)^2 + y^2 = k^2$

$\Rightarrow\ y\dfrac{dy}{dx} = \pm\sqrt{k^2 - y^2}$

which is the required differential equation of given curve. On solving this D.E., we get the Eqn. of curve as follows

$$\int \frac{ydy}{\sqrt{k^2 - y^2}} = \int \pm dx \Rightarrow -\frac{1}{2}.2\sqrt{k^2 - y^2} = \pm x + c$$

$-\sqrt{k^2 - y^2} = \pm x + c$

Since, it passes through $(0, k)$, we get $c = 0$

$\therefore$ Equation of curve is

$-\sqrt{k^2 - y^2} = \pm x \Rightarrow x^2 + y^2 = k^2$

7. $(a + bx)e^{\frac{y}{x}} = x$

$\Rightarrow\ e^{\frac{y}{x}} = \dfrac{x}{a + bx}$ ...(i)

Diff. w.r. to x, we get

$$e^{\frac{y}{x}}\frac{\left[(x)\frac{dy}{dx} - y\right]}{x^2} = \frac{a + bx - bx}{(a + bx)^2}$$

$\Rightarrow\ \left(x\dfrac{dy}{dx} - y\right)e^{\frac{y}{x}} = \dfrac{ax^2}{(a + bx)^2}$ ...(ii)

From (i) and (ii) we get

$\left(x\dfrac{dy}{dx} - y\right) = \dfrac{ax}{a + bx}$ ...(iii)

Differentiating (iii) w. r. to x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-\frac{dy}{dx}=\frac{(a+bx)a-axb}{(a+bx)^2}$$

$$\Rightarrow\ x\frac{d^2y}{dx^2}=\frac{a^2}{(a+bx)^2}$$

$$\Rightarrow\ x^3\frac{d^2y}{dx^2}=\left(\frac{ax}{a+bx}\right)^2 \quad ...(iv)$$

Comparing (iii) and (iv), we get

$$x^3\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^2$$

Topic-2: General & Particular Solution of Differential Equation

1. (a) Given differential equation can be written as

$$\int dy=\int\frac{1}{\left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)\left(\sqrt{9+\sqrt{x}}\right)8\sqrt{x}}dx$$

Putting $\sqrt{4+\sqrt{9+\sqrt{x}}}=t$, we get

$$\frac{1}{2\sqrt{4+\sqrt{9+\sqrt{x}}}\cdot 2\sqrt{9+\sqrt{x}}\cdot 2\sqrt{x}}dx=dt$$

$$\therefore\ \int dy=\int dt\ \Rightarrow\ y=t+c$$

$$\Rightarrow\ y=\sqrt{4+\sqrt{9+\sqrt{x}}}+c$$

Now, $y(0)=\sqrt{7}\ \Rightarrow\ c=0$

$$\therefore\ y=\sqrt{4+\sqrt{9+\sqrt{x}}}\Rightarrow y(256)=3$$

2. (c) $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{y}\ \Rightarrow\ \frac{-2y}{\sqrt{1-y^2}}dy+2dx=0$

$$\Rightarrow\ \sqrt{1-y^2}+x=c\ \Rightarrow\ (x-c)^2+y^2=1$$

which is a circle of fixed radius 1 and variable centre $(c, 0)$ lying on x-axis.

3. (a) $ydx+y^2dy=x\,dy\ ;x\in R, y>0, y(1)=1$

$$\Rightarrow\ \frac{ydx-xdy}{y^2}+dy=0\ \Rightarrow\ \frac{d}{dx}\left(\frac{x}{y}\right)+dy=0$$

On integration, we get $\frac{x}{y}+y=c$

$y(1)=1\ \Rightarrow\ c=2,\ \therefore\ \frac{x}{y}+y=2$

Now to find $y(-3)$, putting $x=-3$ in above equation, we get

$$-\frac{3}{y}+y=2\Rightarrow y^2-2y-3=0\Rightarrow y=3,-1$$

$\because\ y>0,\ \therefore\ y=3$

4. (c) The given differential equation is $(x^2+y^2)dy=xy\,dx$ such that $y(1)=1$ and $y(x_0)=e$

$$\Rightarrow\ \frac{dy}{dx}=\frac{xy}{x^2+y^2}$$

Put $y=vx,\ \therefore\ v+x\frac{dv}{dx}=\frac{v}{1+v^2}$

$$\Rightarrow\ x\frac{dv}{dx}=\frac{-v^3}{1+v^2}\Rightarrow\int\frac{1+v^2}{v^3}dv+\int\frac{dx}{x}=0$$

$$\Rightarrow\ -\frac{1}{2v^2}+\log|v|+\log|x|=c$$

$$\Rightarrow\ \log y=c+\frac{x^2}{2y^2}\qquad(\because v=y/x)$$

Now, $y(1)=1\ \Rightarrow\ c=-\frac{1}{2},\ \therefore\ \log y=\frac{x^2-y^2}{2y^2}$

since $y(x_0)=e\ \Rightarrow\ \log e=\frac{x_0^2-e^2}{2e^2}\ \Rightarrow\ x_0=\sqrt{3}e$

5. (c) Given that $y=y(x)$

and $x\cos y+y\cos x=\pi$ ...(i)

From eqn. (i), when $x=0$ then $y=\pi$

On differentiating (i) with respect to x, we get

$$-x\sin y.y'+\cos y+y'\cos x-y\sin x=0$$

$$\Rightarrow\ y'=\frac{y\sin x-\cos y}{\cos x-x\sin y}\quad ...(ii)$$

$$\Rightarrow\ y'(0)=1$$

(using $y(0)=\pi$)

Differentiating (ii) with respect to x, we get

$$y''=\frac{(y'\sin x+y\cos x+\sin y.y')(\cos x-x\sin y)-(-\sin x-\sin y-x\cos y y')(y\sin x-\cos y)}{(\cos x-x\sin y)^2}$$

$$\Rightarrow\ y''(0)=\frac{\pi(1)-1}{1}=\pi-1$$

6. (a) $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx}=-\cos x, y(0)=1$

$$\Rightarrow\ \frac{dy}{(1+y)}=\frac{-\cos x}{2+\sin x}dx$$

On integrating both sides, we get

$$\Rightarrow\ \ln(1+y)=-\ln(2+\sin x)+c \quad ...(i)$$

Put $x=0$ and $y=1\ \Rightarrow\ \ln 2=-\ln 2+c\ \Rightarrow\ c=\ln 4$

Put $x=\frac{\pi}{2}$ in eqn. (i), $\ln(1+y)=-\ln 3+\ln 4=\ln\frac{4}{3}$

$$\Rightarrow\ y=\frac{1}{3}$$

7. **(a)** $\frac{dy}{dt}-\frac{t}{1+t}y=\frac{1}{1+t}$

I.F. $=e^{-\int\frac{t}{1+t}dt}=e^{-\int\left(1-\frac{1}{1+t}\right)dt}=e^{-(t-\log(1+t))}$

$=e^{-t}.e^{\log(1+t)}=(1+t)e^{-t}$

Hence solution is

$y.e^{-t}(1+t)=\int\frac{1}{(1+t)}e^{-t}(1+t)\,dt+c$

$\Rightarrow\ y.e^{-t}(1+t)=-e^{-t}+c \Rightarrow y=-\frac{1}{1+t}+\frac{ce^{t}}{1+t}$

$\because\ y(0)=-1,\ \therefore\ c=0$

Hence, $y=-\frac{1}{1+t}\ \Rightarrow\ y(1)=-\frac{1}{2}$

8. **(c)** Given differential equation is

$\left(\frac{dy}{dx}\right)^2-x\frac{dy}{dx}+y=0$ ...(i)

(a) $y=2\Rightarrow\frac{dy}{dx}=0$

On putting in Eq. (i)

$0^2-x(0)+y=0$

$\Rightarrow$ y = 0 which is not satisfied.

(b) $y=2x\Rightarrow\frac{dy}{dx}=2$

On putting in Eq. (i),

$(2)^2-x.2+y=0$

$\Rightarrow 4-2x+y=0$

$\Rightarrow y=2x$ which is not satisfied.

(c) $y=2x-4\Rightarrow\frac{dy}{dx}=2$

On putting in Eq. (i)

$(2)^2-x-2+y$

$4-2x+2x-4=0$ $[\because y=2x-4]$

y = 2x – 4 is satisfied.

(d) $y=2x^2-4$

$\frac{dy}{dx}=4x$

On putting in Eq. (i),

$(4x)^2-x.4x+y=0$

$\Rightarrow y=0$ which is not satisfied.

9. **(8)** Given differential equation $x\,dy-(y^2-4y)\,dx=0$

$\frac{dy}{y^2-4y}=\frac{dx}{x}$

$\Rightarrow\ \frac{1}{4}\int\left(\frac{1}{y-4}-\frac{1}{y}\right)dy=\int\frac{dx}{x}$

$\Rightarrow\frac{1}{4}\ln\left|\frac{y-4}{y}\right|=\ln|x|+c$

$\Rightarrow$ Put $x=1$ and $y=2$, we get $c=0$

$\ln\left|\frac{y-4}{y}\right|=4\ln|x|$

so, $\frac{y-4}{y}=\pm x^4$ or, $y=\frac{4}{1\pm x^4}$

When $y=\frac{4}{1-x^4}$ then $y(1)$ is not define.

Take $y=\frac{4}{1+x^4}\ \Rightarrow\ y\left(\sqrt{2}\right)=\frac{4}{5}$

So, $10y\left(\sqrt{2}\right)=8$

10. **(2)** $f(x+y)=f(x)f'(y)+f'(x)f(y)$ ...(i)

On putting x = y = 0, we get

$f(0)=2f'(0)f(0)\ \Rightarrow\ f'(0)=\frac{1}{2}$ $[\because f(0)=1]$

On putting y = 0 in equation (i), we get

$f(x)=f(x)\ f'(0)+f'(x)\ f(0)$

$\Rightarrow\ f'(x)=\frac{f(x)}{2}\ \Rightarrow\ \int\frac{f'(x)}{f(x)}dx=\frac{1}{2}\int dx$

$[\because f(0)=1$ and $f'(0)=\frac{1}{2}]$

$\Rightarrow\ \log_e f(x)=\frac{x}{2}+\log_e c$

$\Rightarrow\ f(x)=ce^{x/2}\ \Rightarrow\ f(x)=e^{x/2}$ $[\because f(0)=1]$

$\Rightarrow\ \log_e(f(x))=\frac{x}{2}\ \Rightarrow\ \log_e(f(4))=2$

11. **(0.4)** $\frac{dy}{dx}=(5y+2)(5y-2)=25\left(y+\frac{2}{5}\right)\left(y-\frac{2}{5}\right)$

$\Rightarrow\ \frac{1}{25}\int\frac{dy}{\left(y+\frac{2}{5}\right)\left(y-\frac{2}{5}\right)}=\int dx$

$\Rightarrow\ \frac{1}{25}\int\frac{5}{4}\left[\frac{1}{y-\frac{2}{5}}-\frac{1}{y+\frac{2}{5}}\right]dy=\int dx$

$\Rightarrow\ \frac{1}{25}\times\frac{5}{4}\ell n\left|\frac{y-\frac{2}{5}}{y+\frac{2}{5}}\right|=x+c$

$\Rightarrow\ \frac{1}{20}\ell n\left|\frac{5y-2}{5y+2}\right|=x+c$

As, f(0) = 0, $\therefore$ 0 = 0 + c $\Rightarrow$ c = 0

Hence, $\left|\frac{5y-2}{5y+2}\right|=e^{20x}$

$\Rightarrow\ \lim_{x\to-\infty}\left|\frac{5f(x)-2}{5f(x)+2}\right|=\lim_{x\to-\infty}e^{20x}=e^{-\infty}=0$

$\Rightarrow\ 5\lim_{x\to-\infty}f(x)-2=0\ \Rightarrow\ \lim_{x\to-\infty}f(x)=\frac{2}{5}=0.4$

12. **(c, d)**

Tangent to the curve y = y (x) at point P (x, y) is given by

$Y-y=y'(x)(X-x)$

It intersects y-axis at Y_p, putting x = 0

$Y-y=-xy'(x)\Rightarrow Y=y-xy'(x)$

$\therefore\ Y_p(0,y-xy'(x))$

Given $PY_p=1\Rightarrow\sqrt{(x-0)^2+(y-y+xy'(x))^2}=1$

$\Rightarrow x^2 + x^2(y'(x))^2 = 1 \Rightarrow y'(x) = \pm\frac{\sqrt{1-x^2}}{x}$

Now y = y (x) less in first quadrant and its tangent passes through (1, 0), therefore it has to be a decreasing function, so derivative should be negative

$\therefore y'(x) = \frac{-\sqrt{1-x^2}}{x}$ $\left[\text{or } xy'(x) + \sqrt{1-x^2} = 0\right]$

$\Rightarrow y(x) = -\int\frac{\sqrt{1-x^2}}{x}dx$

put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$y = -\int\frac{\cos\theta}{\sin\theta}\cos\theta\, d\theta = -\int(\text{cosec}\,\theta - \sin\theta)\, d\theta$

$y = +\log|\text{cosec}\,\theta + \cot\theta| - \cos\theta + c$

$y = \log\left|\frac{1+\sqrt{1-x^2}}{x}\right| - \sqrt{1-x^2} + c$

for x = 1 and y = 0, we get c = 0

$y = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

∴ options (c) and (d) are correct.

13. **(a, d)** $[(x+2)(x+2+y)]\frac{dy}{dx} - y^2 = 0$, Put $y = (x+2)t$

$\Rightarrow \frac{dy}{dx} = (x+2)\frac{dt}{dx} + t$

$(x+2)^2 = 0$ or $(1+t)\left((x+2)\frac{dt}{dx} + t\right) - t^2 = 0$

$(x+2)(1+t)\frac{dt}{dx} + t = 0$

$\left(\frac{1+t}{t}\right)dt = -\frac{dx}{x+2}$

$\ln t + t = -\ln(x+2) + c$

$\Rightarrow \ln\left(\frac{y}{x+2}\right) + \left(\frac{y}{x+2}\right)$

$= -\ln(x+2) + c$

$\ln y - \ln(x+2) + \frac{y}{x+2} = -\ln(x+2) + c$

$\ln y + \frac{y}{x+2} = c$

$\ln 3 + 1 = c \Rightarrow \ln y + \frac{y}{x+2} = \ln 3e$

(A) $\ln y + \frac{y}{x+2} = \ln 3e = \ln(x+2) + 1 = \ln 3 + 1$

⇒ one solution

(D) $y = (x+3)^2 \Rightarrow \ln(x+3)^2 + \frac{(x+3)^2}{x+2} = \ln 3 + 1$

$2\ln(x+3) + \frac{(x+2)^2 + 1 + 2(x+2)}{x+2} = \ln 3 + 1$

$g(x) = 2\ln(x+3) + (x+2) + 2 + \frac{1}{(x+2)} - \ln 3 - 1$

$g(x) = \frac{2}{(x+3)} + 1 + 0 - \frac{1}{(x+2)^2} = \frac{2(x+2)^2 - (x+3)}{(x+3)(x+2)^2} + 1 > 0$

Since x > 0 given and g(0) > 0, therefore g(x) will never intersect x-axis when x > 0.

14. **(a, c)** $\frac{dy}{dx} + \frac{e^x}{1+e^x}y = \frac{1}{1+e^x}$

I.F. = $1 + e^x$. Hence solution is

$y(1+e^x) = x + c$

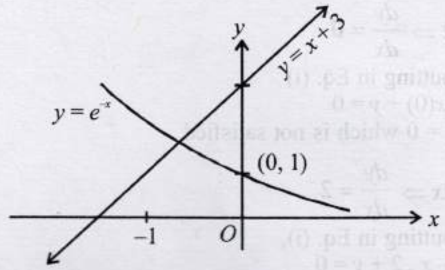
Now, $y(0) = 2 \Rightarrow c = 4$, $\therefore y = \frac{x+4}{e^x+1} \Rightarrow y(-4) = 0$

Also $\frac{dy}{dx} = \frac{(e^x+1) - e^x(x+4)}{(e^x+1)^2}$

For critical point, put $\frac{dy}{dx} = 0$

$\Rightarrow e^x(x+3) = 1 \Rightarrow x + 3 = e^{-x}$

Its solution will be intersection point of $y = x + 3$ and $y = e^{-x}$



Clearly there is a critical point in (–1, 0).

15. **(c)** The given differential equation is

$x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$

$\Rightarrow \int\frac{dy}{y\sqrt{y^2-1}} = \int\frac{dx}{x\sqrt{x^2-1}} \Rightarrow \sec^{-1}y = \sec^{-1}x + c$

$\Rightarrow y = \sec[\sec^{-1}x + c]$, $\because y(2) = \frac{2}{\sqrt{3}}$

$\Rightarrow \frac{2}{\sqrt{3}} = \sec(\sec^{-1}2 + c) \Rightarrow \sec^{-1}\frac{2}{\sqrt{3}} - \sec^{-1}2 = c$

$\Rightarrow c = \frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{6}$, $\therefore y = \sec\left[\sec^{-1}x - \frac{\pi}{6}\right]$

∴ Statement 1 is true.

Also $\frac{1}{y} = \cos\left[\cos^{-1}\frac{1}{x} - \frac{\pi}{6}\right]$

$= \cos\left(\cos^{-1}\frac{1}{x}\right)\cos\frac{\pi}{6} + \sin\left(\cos^{-1}\frac{1}{x}\right)\sin\frac{\pi}{6}$

$\Rightarrow \frac{1}{y} = \frac{\sqrt{3}}{2x} + \frac{1}{2}\sqrt{1 - \frac{1}{x^2}}$

∴ Statement 2 is false.

16. We know that length of tangent to curve $y = f(x)$ is given by

$$\left|\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)}\right|$$

According to the question,

$$\left|\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)}\right| = 1 \Rightarrow y^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\} = \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{y^2}{1-y^2} \Rightarrow \frac{dy}{dx} = \pm\frac{y}{\sqrt{1-y^2}}$$

$$\Rightarrow \int\frac{\sqrt{1-y^2}}{y}dy = \pm\int dx$$

Put $y = \sin\theta$ so that $dy = \cos\theta\, d\theta$

$$\therefore \int\frac{\cos\theta}{\sin\theta}\cos\theta\, d\theta = \pm x + c$$

$$\Rightarrow \int(\operatorname{cosec}\theta - \sin\theta)d\theta = \pm x + c$$

$$\Rightarrow \log|\operatorname{cosec}\theta - \cot\theta| + \cos\theta = \pm x + c$$

$$\Rightarrow \log\left|\frac{1-\sqrt{1-y^2}}{y}\right| + \sqrt{1-y^2} = \pm x + c$$

17. Let the water level be at a height h after time t, and water level falls by dh in time dt and the corresponding volume of water gone out be dV.

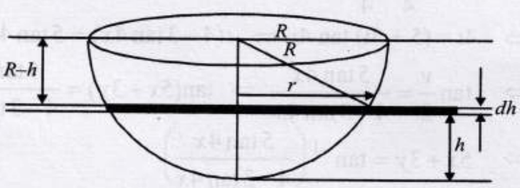
$$\Rightarrow |dV| = |\pi r^2 dh| \qquad (\because dh \text{ is very small})$$

$$\Rightarrow \frac{dV}{dt} = -\pi r^2\frac{dh}{dt} \qquad (\because \text{ as } t \text{ increases, } h \text{ decreases})$$

Now, velocity of water, $v = \frac{3}{5}\sqrt{2gh}$

Rate of flow of water $= Av\,(A = 12\text{ cm}^2)$

$$\Rightarrow \frac{dV}{dt} = \left(\frac{3}{5}\sqrt{2gh}A\right) = -\pi r^2\frac{dh}{dt}$$



Also from figure,

$$R^2 = (R-h)^2 + r^2 \Rightarrow r^2 = 2hR - h^2$$

So, $\frac{3}{5}\sqrt{2g}.\sqrt{h}A = -\pi(2hR-h^2).\frac{dh}{dt}$

$$\Rightarrow \frac{2hR-h^2}{\sqrt{h}}dh = -\frac{3}{5\pi}\sqrt{2g}.A.dt$$

Integrating, $\int_R^0(2R\sqrt{h}-h^{3/2})dh = -\frac{3\sqrt{2g}}{5\pi}.A.\int_0^T dt$

$$\Rightarrow T = \frac{-5\pi}{3A\sqrt{2g}}\left(2R\cdot\frac{h^{\frac{3}{2}}}{\frac{3}{2}} - \frac{h^{\frac{5}{2}}}{\frac{5}{2}}\right)_R^0$$

$$= \frac{5\pi}{3A\sqrt{2g}}\left(-\frac{2}{5}R^{5/2} + \frac{4R}{3}.R^{3/2}\right) = \frac{5\pi}{3A\sqrt{2g}}.\frac{14}{15}.R^{5/2}$$

$$= \frac{5\pi}{3\times 12\times\sqrt{2}\sqrt{g}}\times\frac{14}{15}\times(200)^{5/2}$$

$$= \frac{7\pi}{54\times\sqrt{2}\sqrt{g}}(2)^{5/2}\times(100)^{5/2}$$

$$= \frac{7\pi}{54\sqrt{2}\sqrt{g}}\times(2)^2\times(2)^{1/2}\times(100)^2\times(100)^{1/2}$$

$$= \frac{14\pi}{27\sqrt{g}}(10)^5 \text{ units.}$$

18. Let X_0 be initial population of the country and Y_0 be its initial food production. Let the average consumption be a unit. Therefore, food required initially aX_0. It is given

$$Y_p = aX_0\left(\frac{90}{100}\right) = 0.9\,aX_0 \qquad \text{...(i)}$$

Let X be the population of the country in year t.

Then, $\frac{dX}{dt}$ = Rate of change of population

$$= \frac{3}{100}X = 0.03X$$

$$\Rightarrow \frac{dX}{X} = 0.03\,dt \Rightarrow \int\frac{dX}{X} = \int 0.03dt$$

$\Rightarrow \log X = 0.03\,t + c$

$\Rightarrow X = A.e^{0.03t}$, where $A = e^c$

At $t = 0$, $X = X_0$, thus $X_0 = A$

$\therefore X = X_0\,e^{0.03t}$

Let Y be the food production in year t.

Then, $Y = Y_0\left(1+\frac{4}{100}\right)^t = 0.9aX_0(1.04)^t$

$\because Y_0 = 0.9\,aX_0$ [from Eq. (i)]

Food consumption in the year t is $aX_0\,e^{0.03t}$

Again, $Y - X \geq 0$ [given]

$\Rightarrow 0.9\,X_0\,a(1.04)^t > aX_0e^{0.03t}$

$$\Rightarrow \frac{(1.04)^t}{e^{0.03t}} > \frac{1}{0.9} = \frac{10}{9}.$$

Taking log on both sides, we get

$t[\log(1.04) - 0.03] \geq \log 10 - \log 9$

$$\Rightarrow t \geq \frac{\log 10 - \log 9}{\log(1.04) - 0.03}$$

Thus, the least integral values of the year n, when the country becomes self-sufficient is the smallest integer greater than or equal to $\frac{\log 10-\log 9}{\log(1.04)-0.03}$.

19. Equation of normal at point (x,y) is

$$Y-y=-\frac{dx}{dy}(X-x) \quad \text{...(i)}$$

Distance of perpendicular from the origin to Eq. (i)

$$=\frac{\left|y+\frac{dx}{dy}\cdot x\right|}{\sqrt{1+\left(\frac{dx}{dy}\right)^2}}$$

Also, distance between P and X-axis is $|y|$.

$$\therefore \quad \frac{\left|y+\frac{dx}{dy}\cdot x\right|}{\sqrt{1+\left(\frac{dx}{dy}\right)^2}}=|y|$$

$$\Rightarrow y^2+\frac{dx}{dy}x^2+2xy\frac{dx}{dy}=y^2\left[1+\left(\frac{dx}{dy}\right)^2\right]$$

$$\Rightarrow \left(\frac{dx}{dy}\right)^2(x^2-y^2)+2xy\frac{dx}{dy}=0$$

$$\Rightarrow \frac{dx}{dy}\left[\left(\frac{dx}{dy}\right)(x^2-y^2)+2xy\right]=0$$

$$\Rightarrow \frac{dx}{dy}=0 \text{ or } \frac{dy}{dx}=\frac{y^2-x^2}{2xy}$$

But $\frac{dx}{dy}=0 \quad \Rightarrow x=c$, where c is a constant.

Since, curve passes through (1, 1), we get the equation of the curve as $x = 1$.

The equation $\frac{dy}{dx}=\frac{y^2-x^2}{2xy}$ is a homogeneous equation.

Put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{v^2x^2-x^2}{2x^2v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2-1}{2v}-v=\frac{v^2-1-2v^2}{2v}=-\frac{v^2+1}{2v}$$

$$\Rightarrow \frac{-2v}{v^2+1}dv=\frac{dx}{x}$$

$$\Rightarrow c_1-\log(v^2+1)=\log|x|$$

$$\Rightarrow \log|x|(v^2+1)=c_1 \Rightarrow |x|\left(\frac{y^2}{x^2}+1\right)=e^{c_1}$$

$\Rightarrow x^2+y^2=\pm e^{c_1}x$ or $x^2+y^2=\pm e^c x$ is passing through (1, 1).

$\therefore 1+1=+e^c.1$

$\Rightarrow \pm e^c=2$

Hence, required curve is $x^2+y^2=2x$.

20. Given, D.E., $\frac{dy}{dx}=\sin(10x+6y)$

Put $10x+6y=v$

$$\therefore \quad 10+6\frac{dy}{dx}=\frac{dv}{dx} \quad \therefore \frac{dv}{dx}-10=6\sin v$$

$$\Rightarrow \frac{dv}{6\sin v+10}=dx \text{ or } \frac{dv}{12\sin\frac{v}{2}\cos\frac{v}{2}+10}=dx$$

On dividing numerator and denominator by $\cos^2\left(\frac{v}{2}\right)$, we get

$$\frac{\sec^2\frac{v}{2}dx}{12\tan\frac{v}{2}+10\sec^2\frac{v}{2}}=dx$$

Now put, $\tan\left(\frac{v}{2}\right)=t \Rightarrow \frac{1}{2}\sec^2\frac{v}{2}=\frac{dt}{dv}$

$$\Rightarrow \quad 2dt=\sec^2\frac{v}{2}dv$$

$$\therefore \quad \frac{2dt}{12t+10(1+t^2)}=dx \text{ or } \frac{dt}{5t^2+6t+5}=dx$$

$$\Rightarrow \frac{dt}{\left(t+\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}=5dx$$

$$\Rightarrow \frac{5}{4}\tan^{-1}\frac{5t+3}{4}=5x+5c$$

$$\Rightarrow \tan^{-1}\frac{5t+3}{4}=4x+c \quad \text{...(i)}$$

At origin $x=0, y=0 \Rightarrow v=0 \Rightarrow t=\tan\frac{v}{2}=0$

$$\therefore \quad \tan^{-1}\frac{3}{4}=c$$

On putting the value of c in equation (i), we get

$$\tan^{-1}\frac{5t+3}{4}-\tan^{-1}\frac{3}{4}=4x$$

$$\Rightarrow \frac{\frac{5t+3}{4}-\frac{3}{4}}{1+\frac{5t+3}{4}\cdot\frac{3}{4}}=\tan 4x \Rightarrow \frac{20t}{25+15t}=\tan 4x$$

$$\Rightarrow 4t=(5+3t)\tan 4x \Rightarrow t(4-3\tan 4x)=5\tan 4x$$

$$\Rightarrow \tan\frac{v}{2}=\frac{5\tan 4x}{4-3\tan 4x} \Rightarrow \tan(5x+3y)=\frac{5\tan 4x}{4-3\tan 4x}$$

$$\Rightarrow 5x+3y=\tan^{-1}\left(\frac{5\tan 4x}{4-3\tan 4x}\right)$$

$$\Rightarrow y=\frac{1}{3}\left(\tan^{-1}\left(\frac{5\tan 4x}{4-3\tan 4x}\right)-5x\right)$$

Topic-3: Linear Differential Equation of First Order

1. (c) Given that $3\int_1^x f(t)dt = xf(x) - \frac{x^3}{3}$

differentiate w.r.t. to x both side

$3f(x) = f(x) + xf'(x) - x^2$

$\frac{dy}{dx} - \left(\frac{2}{x}\right)y = x$ [∵ y = f(x)]

$IF = e^{-2\ell nx} = \frac{1}{x^2} \Rightarrow y\left(\frac{1}{x^2}\right) = \int x.\frac{1}{x^2}dx$

$y = x^2\,\ell nx + cx^2;\ \therefore\ y(1) = \frac{1}{3} \Rightarrow c = \frac{1}{3}$

Now, $y(e) = \frac{4e^2}{3}$

2. (b) $\frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{x^4+2x}{\sqrt{1-x^2}}$

I.F. $e^{\int\frac{-x}{1-x^2}dx} = e^{\frac{+1}{2}\log(1-x^2)} = \sqrt{1-x^2}$

Hence, solution is given by

$y\sqrt{1-x^2} = \int\sqrt{1-x^2}.\frac{x^4+2x}{\sqrt{1-x^2}}dx$

$y\sqrt{1-x^2} = \frac{x^5}{5} + x^2 + c$

$f(0) = 0 \Rightarrow$ At $x = 0, y = 0 \Rightarrow c = 0$

$\therefore\quad y\sqrt{1-x^2} = \frac{x^5}{5} + x^2$

$\Rightarrow\quad y = f(x) = \frac{\frac{x^5}{5}+x^2}{\sqrt{1-x^2}},\ \therefore\ I = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\frac{\frac{x^5}{5}+x^2}{\sqrt{1-x^2}}dx$

$= 2\int_0^{\frac{\sqrt{3}}{2}}\frac{x^2}{\sqrt{1-x^2}}dx \quad \left(\because \frac{x^5}{\sqrt{1-x^2}}\text{ is odd}\right)$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$\therefore\quad I = 2\int_0^{\frac{\pi}{3}}\sin^2\theta\, d\theta = \int_0^{\frac{\pi}{3}}(1-\cos 2\theta)d\theta$

$= \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

3. (16) Given differential equation is $(x^2-5)\frac{dy}{dx} - 2xy = -2x(x^2-5)^2$

$\frac{dy}{dx} - \frac{2x}{x^2-5}y = -2x(x^2-5)$

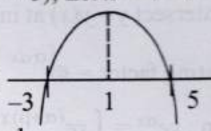
$IF = e^{-\int\frac{2x}{x^2-5}dx} = \frac{1}{(x^2-5)};\ y.\frac{1}{x^2-5} = \int -2x.dx + c$

$\Rightarrow \frac{y}{x^2-5} = -x^2 + c$

put $x = 2, y = 7 \Rightarrow \frac{7}{-1} = -4 + c \Rightarrow c = -3$

$\because y = -(x^2-5)(x^2+3)$; Let $x^2 = t > 0$

$y = -(t-5)(t+3)$



$y_{max} = 16$ when $x^2 = 1$

4. (0) The given equation is $\frac{dy}{dx} + g'(x)y = g(x)\,g'(x)$

I.F. $= e^{\int g'(x)dx} = e^{g(x)}$

Hence solution is $y.e^{g(x)} = \int e^{g(x)}g(x).g'(x)dx$

Put $g(x) = t$ so that $g'(x)\,dx = dt$

$y.e^{g(x)} = \int e^t t\,dt = e^t(t-1) + c$

$\therefore\quad y.e^{g(x)} = e^{g(x)}[g(x)-1] + c$

As $y(0) = 0$ and $g(0) = 0$, $\therefore\ c = 1$

$\therefore\quad y.e^{g(x)} = e^{g(x)}[g(x)-1] + 1$

As $g(2) = 0$, putting $x = 2$ we get

$y(2).e^{g(2)} = e^{g(2)}[g(2)-1] + 1 \Rightarrow y(2) = 0$

5. (c) $\frac{dy}{dx} + 12y = \cos\left(\frac{\pi x}{12}\right)$

I.F. $= e^{12\int dx} = e^{12x}$

Solution is: $y.e^{12x} = \int e^{12x}.\cos\left(\frac{\pi x}{12}\right)dx + C$

$\Rightarrow y.e^{12x} = \frac{e^{12x}}{12^2 + \left(\frac{\pi}{12}\right)^2}\left[12\cos\frac{\pi x}{12} + \frac{\pi}{12}\sin\left(\frac{\pi x}{12}\right)\right] + c$

Put $x = 0, y = 0$ we get

$C = -\frac{12}{12^2 + \left(\frac{\pi}{12}\right)^2}$

So $y = \frac{1}{\lambda}\left[\underbrace{12\cos\left(\frac{\pi x}{12}\right) + \frac{\pi}{12}\sin\left(\frac{\pi x}{12}\right)}_{f_1(x)} - 12e^{-12x}\right]$

$\left[\text{Let}\ \frac{1}{\lambda} = \frac{1}{\left(\frac{\pi}{12}\right)^2 + 12^2}\right]$

$\frac{dy}{dx} = \frac{1}{\lambda}\left[\underbrace{-\pi\sin\left(\frac{\pi x}{12}\right) + \frac{\pi^2}{12^2}\cos\frac{\pi x}{12}}_{f_2(x)} + 144e^{-12x}\right]$

When x is large then $12e^{-12x}$ tends to zero.

But $f_2(x)$ varies in $\left[-\sqrt{\pi^2 + \left(\frac{\pi}{12}\right)^4}, \sqrt{\pi^2 + \left(\frac{\pi}{12}\right)^4}\right]$

Hence $\frac{dy}{dx}$ is changing its sign.

So, $y(x)$ is non-monotonic for all real number.

Also when x is very large then again $-12e^{-12x}$ is almost zero but $f_1(x)$ is periodic, so there exist some β for which
y = β intersect y = y(x) at infinitely many points.

6. **(a, c)** Integrating factor $= e^{\int \alpha\,dx} = e^{\alpha x}$

Solution : $ye^{\alpha x} = \int xe^{(\alpha+\beta)x}\,dx$

Case I :

If $\alpha+\beta = 0$ then $ye^{\alpha x} = \frac{x^2}{2} + C$

Put $x = 1$ and $y = 1$

$\Rightarrow \quad C = e^{\alpha} - \frac{1}{2}$

So, $ye^{\alpha x} = \frac{x^2}{2} + e^{\alpha} - \frac{1}{2}$

$\Rightarrow \quad y = \frac{x^2}{2}.e^{-\alpha x} + \left(e^{\alpha} - \frac{1}{2}\right)e^{-\alpha x}$

for $\alpha = 1$

$y = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$

option (a) is correct.

Case II :

If $\alpha+\beta \neq 0$

$ye^{\alpha x} = \frac{x.e^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{1}{\alpha+\beta}\int e^{(\alpha+\beta)x}dx$

$\Rightarrow \quad ye^{\alpha x} = \frac{xe^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2} + C$

Put $x = 1$ and $y = 1$, we get

$c = e^{\alpha} - \frac{e^{\alpha+\beta}}{\alpha+\beta} + \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}$

$y = \frac{e^{\beta x}}{(\alpha+\beta)^2}((\alpha+\beta)x - 1)$

$\qquad + e^{-\alpha x}\left(e^{x} - \frac{e^{\alpha+\beta}}{\alpha+\beta} + \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}\right)$

For $\alpha = \beta = 1$

$y = \frac{e^x}{4}(2x-1) + e^{-x}\left(e - \frac{e^2}{2} + \frac{e^2}{4}\right)$

$y = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + e^{-x}\left(e - \frac{e^2}{4}\right)$

So, option (c) is correct.

7. **(b, c)** $f(x) = 1 - 2x + \int_0^x e^{x-t}f(t)\,dt$

$\Rightarrow \quad f(x) = 1 - 2x + e^x\int_0^x e^{-t}f(t)dt$

$\Rightarrow \quad f'(x) = -2 + e^x\int_0^x e^{-t}f(t)\,dt + e^x[e^{-x}f(x)]$

$\Rightarrow \quad f'(x) = -2 + [f(x) - 1 + 2x] + f(x)$

$\Rightarrow \quad f'(x) - 2f(x) = 2x - 3$

Its a linear differential equation.

$IF = e^{\int -2dx} = e^{-2x}$

Solution: $f(x) \times e^{-2x} = \int e^{-2x}(2x-3)\,dx$

$\Rightarrow \quad f(x) \times e^{-2x} = \frac{e^{-2x}}{-2}(2x-3) - \int \frac{e^{-2x}}{-2} \times 2\,dx$

$\Rightarrow \quad e^{-2x}f(x) = \frac{e^{-2x}}{-2}(2x-3) + \frac{e^{-2x}}{-2} + c$

$\Rightarrow \quad f(x) = -x + \frac{3}{2} + \frac{1}{-2} + ce^{2x} \Rightarrow f(x) = -x + 1 + ce^{2x}$

From definition of function, f(0) = 1

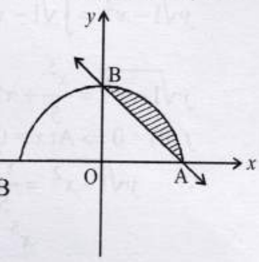
$\therefore$ 1 = 1 + c $\Rightarrow$ c = 0, $\therefore$ f(x) = 1 – x

Clearly curve y = 1 – x, does not pass through (1, 2) but it passes through (2, – 1)

$\therefore$ (a) is false and (b) is true.

Also the area of the region

$1 - x \le y \le \sqrt{1-x^2}$, is shown in the figure by the shaded region, is given by



= Area of quadrant – Area ΔOAB

$= \frac{1}{4} \times \pi \times 1^2 - \frac{1}{2} \times 1 \times 1 = \frac{\pi - 2}{4}$

$\therefore$ (c) is true and (d) is false.

8. **(a)** $f'(x) = 2 - \frac{f(x)}{x} \Rightarrow f'(x) + \frac{1}{x}f(x) = 2$

I.F. $= e^{\log x} = x$, $\therefore$ $f(x).x = \int 2x\,dx = x^2 + c$

$\Rightarrow \quad f(x) = x + \frac{c}{x}, c \neq 0$ as $f(x) \neq 1$

(a) $\lim_{x\to 0^+} f'\left(\frac{1}{x}\right) = \lim_{x\to 0^+}(1 - cx^2) = 1$

(b) $\lim_{x\to 0^+} xf\left(\frac{1}{x}\right) = \lim_{x\to 0^+} x\left(\frac{1}{x} + cx\right) \lim_{x\to 0^+} 1 + cx^2 = 1$

(c) $\lim_{x\to 0^+} x^2f'x = \lim_{x\to 0^+} x^2\left(1 - \frac{c}{x^2}\right) = \lim_{x\to 0^+} x^2 - c = -c$

(d) For $c \neq 0$, f(x) is unbounded as $0 < x < 2$

$\Rightarrow \frac{c}{2} < \frac{c}{x} < \infty \Rightarrow \frac{c}{2} < x + \frac{c}{x} < \infty$

9. **(a, d)** $\frac{dy}{dx} - y\tan x = 2x\sec x,$

I.F. $= e^{-\int \tan x\,dx} = \cos x$

$\therefore \quad y.\cos x = \int 2x\,dx = x^2 + c$

Now, $y(0) = 0 \Rightarrow c = 0, \therefore y = x^2 \sec x$

$\Rightarrow y' = 2x \sec x + x^2 \sec x \tan x$

Now, $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16} \times \sqrt{2} = \frac{\pi^2}{8\sqrt{2}}$

$y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9} \times 2 = \frac{2\pi^2}{9}$

$y'\left(\frac{\pi}{4}\right) = \frac{2\pi}{4} \times \sqrt{2} + \frac{\pi^2}{8\sqrt{2}} \times 1 = \frac{\pi^2}{8\sqrt{2}} + \frac{\pi}{\sqrt{2}}$

$y'\left(\frac{\pi}{3}\right) = \frac{2\pi}{3} \times 2 + \frac{2\pi^2}{9} \times \sqrt{3} = \frac{2\pi^2}{3\sqrt{3}} + \frac{4\pi}{3}$

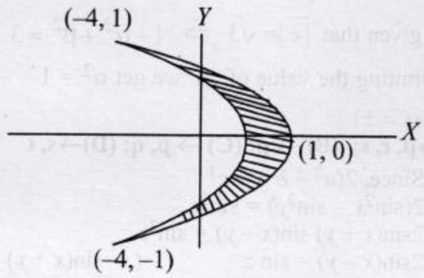
10. (A) → p, (B) → s, (C) → p, (D) → r

(A) $\int_0^{\pi/2} (\sin x)^{\cos x} (\cos x \cot x - \log(\sin x)^{\sin x}) dx$

$= \int_0^1 du$ where $(\sin x)^{\cos x} = u = 1$

(A) → (p)

(B)



Solving $y^2 = -\frac{1}{4}x$ and $y^2 = -\frac{1}{5}(x-1)$, we get intersection points as $(-4, \pm 1)$

$\therefore$ Required area

$= \int_{-1}^{1} [(1 - 5y^2) + 4y^2] dy = 2\int_0^1 (1 - y^2) dy = \frac{4}{3}$,

(B) → (s)

(C) By inspection, the point of intersection of two curves $y = 3^{x-1} \log x$ and $y = x^x - 1$ is (1, 0)

For first curve $\frac{dy}{dx} = \frac{3^{x-1}}{x} + 3^{x-1} \log 3 \log x$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = 1 = m_1$

For second curve $\frac{dy}{dx} = x^x (1 + \log x)$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = 1 = m_2$

$\because m_1 = m_2 \Rightarrow$ Two curves touch each other

$\Rightarrow$ Angle between them is 0°

$\therefore \cos\theta = 1$,

(C) → (p)

(D) $\frac{dy}{dx} = \frac{6}{x+y} \Rightarrow \frac{dx}{dy} - \frac{1}{6}x = \frac{y}{6}$

I.F. $= e^{-y/6}$

$\Rightarrow$ Solution is $x \cdot I \cdot F \cdot = \int \left(\frac{y}{6} \cdot I \cdot F \cdot\right) dy + c$

$\Rightarrow xe^{-y/6} = -ye^{-y/6} - 6e^{-y/6} + c$

$e^{-y/6}(x + y + 6) = c$

$\Rightarrow x + y + 6 = ce^{y/6}$

$\Rightarrow x + y + 6 = 6e^{y/6}$ $\quad \therefore (y(0) = 0)$

$\Rightarrow 12 = 6e^{y/6}$ $\quad$ (using $x + y = 6$)

$\Rightarrow y = 6 \ln 2$ (D) → (r)